AD-A039 208 INDIANA UNIV BLOOMINGTON DEPT OF MATHEMATICS F/G 12/1

ASYMPTOTIC NORMALITY AND CONVERGENCE RATES OF LINEAR RANK STATI--ETC(U)

1977 M L PURI, N S RAJARAM AF-AFOSR-2927-76

UNCLASSIFIED AFOSR-TR-77-0451 NL

1 OF 1

ADA039 208

END

DATE

FILMED

6-77

AFOSR - TR - 77 - 0451

AD A039208

Asymptotic Normality and Convergence Rates of Linear Rank Statistics Under Alternatives

By Madan L. Puri and Navaratna S. Rajaram
Indiana University and Kent State University

0. Summary. Asymptotic normality of linear rank statistics under alternatives is proved employing techniques of Chernoff and Savage, ~~(1958)~~. Under suitable assumptions the rate of convergence to normality under alternatives is also obtained for such statistics. The results of the paper are related to those of Hájek ~~(1968)~~ and Hoeffding, ~~(1973)~~. Results on the rate of convergence extend those of Jurečkova and Puri, (1975), and Bergström and Puri, (1976).

1. Introduction. Let $\{X_{Ni}, i \geq 1\}$ be a sequence of independent random variables with continuous cdfs (cumulative distribution functions) $\{F_{Ni}, i \geq 1\}$ respectively. Consider a linear rank statistic S_N given by

$$S_N = \sum_{i=1}^{N} c_{Ni}\, a_N(R_{Ni}) \tag{1.1}$$

where R_{Ni} is the rank of X_{Ni} in $(X_{N1}, \ldots, X_{NN})$, $(c_{N1}, \ldots, c_{NN})$ are known (regression) constants, and $a_N(1), \ldots, a_N(N)$ are "scores" generated by a known real-valued function $\varphi(t)$, $0 < t < 1$ in either of the following ways:

$$a_N(i) = \varphi(i/(N+1)),\quad 1 \leq i \leq N \tag{1.2}$$

$$a_N(i) = E\,\varphi(U_N^{(i)}),\quad 1 \leq i \leq N \tag{1.3}$$

where $U_N^{(i)}$ is the ith order statistic in a sample of size N from the rectangular distribution over (0,1).

Work supported by the Air Force Office of Scientific Research, A.F.S.C.U.S.A.F., Grant No. AFOSR76-2927. Reproduction in whole or part permitted for any purpose of the United States Government.

See form 1473

AIR FORCE OFFICE OF SCIENTIFIC RESEARCH (AFSC)
NOTICE OF TRANSMITTAL TO DDC
This technical report has been reviewed and is approved for public release IAW AFR 190-12 (7b).
Distribution is unlimited.
A. D. BLOSE
Technical Information Officer

We make the following assumptions:

(I A) $$\max_{1\le i\le N} |c_{Ni}|/s_N = O(N^{-\frac{1}{2}})$$

(I B) $$|\varphi^{(i)}(t)| \le K[t(1-t)]^{\delta - i - \frac{1}{2}}, \quad i=0,1;\ \delta>0,$$

K a generic constant.

s_N^2 is approximate variance of S_N and is given by (1.7) and (1.8) below.

Our main results are the following:

<u>Theorem 1.1.</u> <u>Let the scores</u> $a_N(i)$, $1\le i\le N$ <u>be defined as in</u> (1.2). <u>Then, under the assumptions</u> (I A) <u>and</u> (I B),

(1.4) $$\sup_x \left| P\left(\frac{S_N - \mu_N}{s_N} \le x\right) - \Phi(x)\right| \to 0 \quad \text{as} \quad N\to\infty,\ s_N \not\to 0$$

<u>where</u>

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt,$$

(1.5) $$\mu_N = \sum_{i=1}^{N} c_{Ni} \int_{-\infty}^{\infty} \varphi(H(x))\, dF_{Ni}(x),$$

(1.6) $$H(x) = \frac{1}{N}\sum_{i=1}^{N} F_{Ni}(x),$$

(1.7) $$s_N^2 = \sum_{i=1}^{N} s_{Ni}^2; \quad s_{Ni}^2 = \mathrm{Var}(A_{Ni}(X_{Ni})),$$

(1.8) $$A_{Ni}(x) = \frac{1}{N}\sum_{j=1}^{N} (c_{Ni} - c_{Nj}) \int \{I_{[x\le y]} - F_{Ni}(y)\}\varphi'(H(y))\, dF_{N}(y),$$

<u>and</u>

$$I_{[x\le y]} = \begin{cases} 0 & \text{if } y < x \\ 1 & \text{if } y \ge x \end{cases}$$

Corollary 1.1. Let $\varphi(t) = F^{-1}(t)$, where F is a cdf. Let the scores $a_N(i)$, $1 \le i \le N$ be given by (1.3). Then, under the assumptions (I A) and (I B), the conclusions of Theorem 1.1 hold.

Theorem 1.2. Let the scores $a_N(i)$, $1 \le i \le N$ be given by (1.2) and the assumption (I A) be satisfied. Let

(1.10) $$\sup_x |\varphi'(x)| = \|\varphi'\| < \infty$$

then

(1.11) $$\sup_x \left| P\left(\frac{S_N - \mu_N}{s_N} \le x\right) - \Phi(x) \right| \le c_1 \ell_{3,N} + \Delta_N$$

where

(1.12) $$\ell_{3,N} = \rho_N^3 / s_N^3 \quad , \; c_1 = 0.7985$$

(1.13) $$\rho_N^3 = \sum_{i=1}^{N} \rho_{Ni}^3 \, , \; \rho_{Ni}^3 = E\,|A_{Ni}(X_{Ni})|^3, \text{ and } \Delta_N \to 0$$

Furthermore,

(1.14) $$N^{\frac{1}{2}} \Delta_N = O_p\left(\|\varphi'\| s_N^{-1} \sum_{i=1}^{N} |c_{Ni}|\right)$$

Remark. Theorem 1.1 has been proved by Hájek (1968). His conditions on the score generating function φ are milder than ours but the conclusion of our theorem 1.1 is sharper in the sense that the centering constant μ_N appears naturally in place of ES_N given by Hájek (1968). Corollary 1.1 is an extension of a similar result proved by Chernoff and Savage (1958) for the two sample problem and serves to remove some of the complications encountered in Hoeffding (1973). Theorem 1.2 is related to the results of Bickel (1972), Jurečkova-Puri (1975), and Bergström-Puri (1976). However, the bounds obtained in these papers are non-random and thus sharper than ours. On the other hand, our conditions on the score generating function φ are milder. We believe that the theorem 1.2 is true even when the condition (1.10) is replaced by the assumption (I B). At the present time, the theory of asymptotic expansion for sums of dependent random variables is still at a rudimentary stage (Stein (1970)) and it is doubtful if the random term Δ_N can be removed without additional assumptions on the underlying distributions.

<u>Asymptotic Normality of</u> S_N. The proof of theorem 1.1 (and corollary 1.2) will be along the lines of the Chernoff-Savage theorem (1958) as given in Puri and Sen (1971) with some modifications necessitated by greater generality of the present problem.

First we introduce the following notations:

(2.1) $$H_N(x) = \frac{1}{N} \sum_{i=1}^{N} I_{[X_{Ni} \le x]}$$

(2.2) $$H(x) = \frac{1}{N} \sum_{i=1}^{N} F_{Ni}(x)$$

(2.3) $$C_N(x) = \sum_{i=1}^{N} c_{Ni} I_{[X_{Ni} \le x]}$$

(2.4) $$C(x) = \sum_{i=1}^{N} c_{Ni} F_{Ni}(x) .$$

Note that the functions $H_N(x)$ and $C_N(x)$ are stochastic variables whereas $H(x)$ and $C(x)$ are non-random though depending on N .

Then the following inequalities are obvious:

(2.5) $$|C_N(x)| \le N \max_{1 \le i \le N} |c_{Ni}| H_N(x)$$

(2.6) $$|C(x)| \le N \max_{1 \le i \le N} |c_{Ni}| H(x) , \quad -\infty < x < \infty .$$

<u>Proof of Theorem 1.1</u>. We rewrite S_N defined in (1.1) as

(2.7) $$S_N = \int_{-\infty}^{\infty} \varphi\left(\frac{N}{N+1} H_N(x)\right) dC_N(x)$$

$$= \mu_N + B_{1N} + B_{2N} + \sum_{i=1}^{3} D_{iN}$$

where μ_N is given by (1.5), and

(2.8) $$B_{1N} = \int_{-\infty}^{\infty} \varphi(H(x))\, d(C_N(x) - C(x)) ,$$

(2.9) $$B_{2N} = \int_{-\infty}^{\infty} (H_N(x) - H(x)\ \varphi'(H(x))\, dC(x) ,$$

(2.10) $$D_{1N} = \frac{-1}{N+1} \int_{-\infty}^{\infty} H_N(x)\ \varphi'(H(x))\, d\, C_N(x),$$

(2.11) $$D_{2N} = \int_{-\infty}^{\infty} (H_N(x) - H(x))\ \varphi'(H(x))\, d(C_N(x) - C(x)) ,$$

(2.12) $$D_{3N} = \int_{-\infty}^{\infty} \left\{\varphi\left(\frac{N}{N+1} H_N(x)\right) - \varphi(H(x)) - \left(\frac{N}{N+1} H_N(x) - H(x)\right) \varphi'(H(x))\right\} dC_N(x) .$$

The proof will be accomplished if we establish the following:

(a) $|\mu_N| < \infty$,

(b) $(B_{1N} + B_{2N})/s_N$ is asymptotically normal,

(c) $D_{iN} = o_p(s_N)$, $i = 1,2,3$.

Proof of (a). Using (2.4) and (2.6), we obtain

(2.13) $$|\mu_N| \le N \max_{1\le i\le N} |C_{Ni}| \int_{-\infty}^{\infty} |\varphi(H(x))|\, dH(x) < \infty, \text{ by assumption (I B)} .$$

Proof of (b). To prove (b), we shall verify the Liapunov condition for B_{1N}/s_n and B_{2N}/s_n . Integrating B_{2N} by parts, we obtain

(2.14) $$B_{2N} = [H_N(x) - H(x)]B^*(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} B^*(x)\, d[H_N(x) - H(x)]$$

where

$$B^*(x) = \int_{x_0}^{x} \varphi'(H(x))\, dC(x)$$

where x_0 is determined arbitrarily such that $H(x_0) > 0$.

We first consider $\int_{-\infty}^{\infty} B^*(x)\, d[H_N(x) - H(x)]$

(2.15) $$\int_{-\infty}^{\infty} B^*(x)\, d[H_N(x) - H(x)] = \sum_{i=1}^{N} \left[B^*(x_{Ni}) - E\, B^*(x_{Ni}) \right] / N$$

We verify the condition $\frac{1}{S_N^{2+\delta'}} \sum_{i=1}^{N} E\left[|B^*(X_{Ni}) - E\, B^*(X_{Ni})| / N \right]^{2+\delta'} \to 0$ as $N \to \infty$ for some $\delta' > 0$. To verify this, it suffices to show that

(2.16) $$\frac{1}{S_N^{2+\delta'}} \sum_{i=1}^{N} E\left[|B^*(X_{Ni})| / N \right]^{2+\delta'} \to 0, \text{ as } N \to \infty .$$

Choose $\delta' > 0$ such that $(2+\delta')(\delta - \frac{1}{2}) > 1$. Then

(2.17) $$\frac{1}{s_N^{2+\delta'}} \sum_{i=1}^{N} E|B^*(X_{Ni})/N|^{2+\delta'} = \frac{1}{s_N^{2+\delta'} \cdot N^{2+\delta'}} \sum_{i=1}^{N} \int_{-\infty}^{\infty} \left| \int_{x_0}^{x} \varphi'(H(y))\, dC(y) \right| dF_{Ni}(x)$$

$$\le \frac{1}{s_N^{2+\delta'}\, N^{2+\delta'}} \sum_{i=1}^{N} N^{2+\delta'} \max_{1 \le i \le N} |C_{Ni}|^{2+\delta'} \int_{-\infty}^{\infty} \left| \int_{x_0}^{x} \varphi'(H(y))\, dH(y) \right|^{2+\delta'} dF_{Ni}(x)$$

$$\le \left\{ \max_{1 \le i \le N} |C_{Ni}| / s_N \right\}^{2+\delta'} \sum_{i=1}^{N} \int_{-\infty}^{\infty} \left\{ |\varphi(H(x)| + |\varphi(H(x_0))| \right\}^{2+\delta'} dF_{Ni}(x) .$$

$$= O(N^{-\delta'/2}) \int_{-\infty}^{\infty} \{ |\varphi(H(x))| + |\varphi(H(x_0))| \}^{2+\delta'} dH(x) .$$

We now show that

$$\frac{\beta(x)}{s_N} \Big|_{-\infty}^{\infty} = o_p(1) , \text{ where } \beta(x) = [H_N(x) - H(x)] B^*(x) \quad \text{(see (2.14))} .$$

We note that

$$\left|\frac{B_{2N}(x)}{s_N}\right| = N^{\frac{1}{2}} \left|H_N(x) - H(x)\right| \cdot \frac{1}{N^{\frac{1}{2}} s_N} \left|\int_{x_0}^{x} \varphi'(H(y))\,dc(y)\right|$$

$$\leq N^{\frac{1}{2}} \left|H_N(x) - H(x)\right| O(1) \left|\int_{x_0}^{x} \varphi'(H(y))\,dH(y)\right|$$

$$\leq k\, N^{\frac{1}{2}} \left|H_N(x) - H(x)\right| \{H(x)(1 - H(x))\}^{\delta-\frac{1}{2}}$$

Now since $\forall\ \varepsilon > 0,\ \delta' > 0,\ \exists\ c(\varepsilon, \delta') \ni$

$$\text{(2.18)} \qquad P\left[\sup_x N^{\frac{1}{2}} \frac{|H_N(x) - H(x)|}{\{H(x)(1-H(x))\}^{\delta'-\frac{1}{2}}} > c(\varepsilon, \delta')\right] < \varepsilon\,,$$

it follows that with probability $> 1 - \varepsilon$,

$$\left|\frac{B_{2N}(x)}{s_N}\right| \leq k\, \{H(x)(1 - H(x))\}^{\delta-\delta'} c(\varepsilon, \delta') \to 0 \quad \text{as} \quad x \to \pm\infty$$

by choosing $\delta' < \delta$.

Thus the Liapunov condition (2.16) is satisfied for B_{2N}/s_N. The verification (of the Liapunov condition) for B_{1N}/s_N is similar, and the same is true for $(B_{1N}+B_{2N})/s_N$ by using the C_r - inequality. This proves (b).

<u>Proof of (c)</u>.

$$\text{(2.19)} \qquad \left|\frac{B_{1N}}{N}\right| \leq \frac{1}{Ns_N} \left|\sum_{i=1}^{N} \varphi'(H(X_{Ni}))c_{Ni}\right| \leq \frac{1}{N} \sum_{i=1}^{N} V_{Ni}$$

where

$$\text{(2.20)} \qquad V_{Ni} = \left|\varphi'(H(X_{Ni}))\frac{c_{Ni}}{s_N}\right|$$

To establish that $\frac{1}{N}\sum_{i=1}^{N} V_{Ni} \to 0$ in probability, it suffices to show that

$$\text{(2.21)} \qquad N^{-\alpha} \sum_{i=1}^{N} E|V_{Ni}|^{\alpha} < \infty \quad \text{for some} \quad 0 < \alpha < 1 \quad \text{(cf. Loéve (1963), page 241).}$$

Taking $\alpha = 2/3$,

$$\frac{1}{N^{2/3}} \sum_{i=1}^{N} E\,|V_{Ni}|^{2/3} \le \frac{K}{N^{2/3}} \max_{1\le i\le N} \left|\frac{C_{Ni}}{s_N}\right|^{2/3} \times$$

$$\times \sum_{i=1}^{N} \int_{-\infty}^{\infty} \{H(x)(1-H(x))\}^{\frac{2\delta}{3}-1} dF_{Ni}(x)$$

$$\le K \int_0^1 \{u(1-u)\}^{\frac{2\delta}{3}-1} du < \infty \quad \text{uniformly in } N .$$

Now consider

$$D_{2N} = \int_{-\infty}^{\infty} (H_N(x) - H(x))\, \varphi'(H(x))\, d(C_N(x) - C(x))$$

Noting (2.18), it follows that with probability $> 1-\varepsilon$,

$$|H_N(x) - H(x)|\ |\varphi'(H(x))| \le \frac{K}{N^{\frac{1}{2}}} C(\varepsilon,\delta')\,\{H(x)(1-H(x))\}^{\delta-\delta'-1}$$

Setting $0 < \delta^* = \delta - \delta'$ Choosing $\delta' < \delta$, it suffices to show that

(2.22) $$\frac{1}{s_N N^{\frac{1}{2}}} \int_{-\infty}^{\infty} \{H(x)(1-H(x))\}^{\delta^*-1} dC_N(x) = \frac{1}{s_N N^{\frac{1}{2}}} \sum_{i=1}^{N} C_{Ni} \times$$

$$\times \{H(X_{Ni})(1-H(X_{Ni}))\}^{\delta^*-1}$$

Setting

(2.23) $$V_{Ni} = \frac{N^{\frac{1}{2}} C_{Ni}}{s_N} \{H(X_{Ni})(1-H(X_{Ni}))\}^{\delta^*-1} ,$$

we have to show that

(2.24) $$\frac{1}{N} \sum_{i=1}^{N} \{V_{Ni} - E\,V_{Ni}\} \to 0 \quad \text{in probability} .$$

This will follow if we show that for some $\alpha > 0$,

(2.25) $$N^{-(1+\alpha)} \sum_{i=1}^{N} E\,|V_{Ni}|^{1+\alpha} \to 0$$

Choose $\alpha > 0$ such that $(1+\alpha)(\delta^* - 1) > -1$ (i.e. $0 < \alpha < \frac{\delta^*}{1-\delta^*}$).
Then

(2.26) $$N^{-(1+\alpha)} \sum_{i=1}^{N} E \mid V_{Ni} \mid^{1+\alpha} \le \frac{1}{N^{\alpha} \cdot N} \left| \frac{N^{\frac{1}{2}} \max_{1\le i\le N} |c_{Ni}|}{s_N} \right|^{1+\alpha} \times$$

$$\times \sum_{i=1}^{N} E \{H(X_{Ni})(1 - H(X_{Ni}))\}^{(1+\alpha)(\delta^*-1)}$$

$$= O(1) \frac{1}{N^{\alpha}} \int_{-\infty}^{\infty} \{H(x)(1 - H(x))\}^{(1+\alpha)(\delta^*-1)} dH(x) \to 0, \text{ as } N\to\infty .$$

This proves (2.25).
Finally, consider

$$D_{3N} = \int_{-\infty}^{\infty} \left\{ \varphi\left(\frac{N}{N+1} H_N(x)\right) - \varphi(H(x)) - \left(\frac{N H_N(x)}{N+1} - H(x)\right)\varphi'(H(x))\right\} dC_N(x)$$

We have to show that $\frac{D_{3N}}{s_N} = o_p(1)$. Denote

(2.27) $$C_{3N} = \frac{D_{3N}}{N^{\frac{1}{2}} s_N} ,$$

and note that

(2.28) $$|C_{3N}| \le O(1) \int_{-\infty}^{\infty} \left|\varphi\left(\frac{N}{N+1} H_N(x)\right) - \varphi(H(x)) - \left(\frac{N H_N(x)}{N+1} - H(x)\right)\right.$$

$$\varphi'(H(x)) \,|dH_N(x), \text{since} \max_{1\le i\le N} \frac{|c_{Ni}|}{s_N N^{\frac{1}{2}}} = O(N^{-1}) .$$

The proof that the right hand side of (2.28) is $o_p(N^{-\frac{1}{2}})$ follows precisely as in Puri and Sen (1971, pages 401-405). Thus the proof of Theorem 1.1 follows.

<u>Proof of Corollary 1.1.</u> Denote

$$\varphi_N(t) = \sum_{i=1}^{[Nt]} a_{N(i)},\ 0 \le t < 1\ ;\ a_N(i) = E\varphi(U_N^{(i)}),\ 1 \le i \le N$$

where $[\alpha]$ is the greatest integer $\le \alpha$; and let

(2.29) $$S_N^* = \int_{-\infty}^{\infty} \varphi_N\left(\frac{N H_N(x)}{N+1}\right) dC_N(x)$$

The proof of the corollary will be accomplished if we show that $(S_N - S_N^*)/s_N \to 0$ in probability, as $N \to \infty$.

Lemma 2.1. Under the hypotheses of theorem 1.1 and Corollary 1.1,

(2.30) $$\lim_{N\to\infty} \varphi_N (t) = \varphi(t)$$

(2.31) $$\left| \int_{-\infty}^{\infty} \{\varphi_N \left(\frac{N}{N+1} H_N(x)\right) - \varphi\left(\frac{N}{N+1} H_N(x)\right)\} dC_N(t) \right| = o_p(s_N) .$$

Proof. The proof of (2.30) is well known (see Puri and Sen (1971), pages 408-409); and so we prove (2-31).

Using (2.5), we obtain

(2.32) $$\left| \int_{-\infty}^{\infty} \left\{\varphi_N\left(\frac{N H_N(x)}{N+1}\right) - \varphi\left(\frac{N H_N(x)}{N+1}\right) dC_N(x) \right| \right.$$

$$\leq \max_{1\leq i\leq N} |c_{Ni}| \sum_{i=1}^{N} \left| \varphi_N\left(\frac{N H_N(X_{Ni})}{N+1}\right) - \varphi\left(\frac{N H_N(X_{Ni})}{N+1}\right) \right|$$

$$= \max_{1\leq i\leq N} |c_{Ni}| \sum_{i=1}^{N} \left| \varphi_N\left(\frac{i}{N+1}\right) - \varphi\left(\frac{i}{N+1}\right) \right|$$

Now by assumption I (A) ,

(2.33) $$\max_{1\leq i\leq N} \left|\frac{c_{Ni}}{s_N}\right| \sum_{i=1}^{N} \left| \varphi_N\left(\frac{i}{N+1}\right) - \varphi\left(\frac{i}{N+1}\right) \right| = O(1) N^{-\frac{1}{2}} \sum_{i=1}^{N} \left| \varphi_N\left(\frac{i}{N+1}\right) - \varphi\left(\frac{i}{N+1}\right) \right|$$

which $\to 0$ as $N \to \infty$ (cf. Puri and Sen (1971), pages 409-411). This proves the Lemma; and hence the corollary 1.1.

To compute the var$(B_{1N} + B_{2N})$, note that

$$B_{1N} = \sum_{i=1}^{N} \varphi(H(X_{Ni})) c_{Ni} + \text{constant} .$$

$$B_{2N} = -\frac{1}{N} \sum_{i=1}^{N} B^*(X_{Ni}) + \text{constant, where}$$

$$B^*(x) = \int_{x_0}^{x} \varphi'(H(y))\, dc(y) = \sum_{j=1}^{N} c_{Nj} \int_{x_0}^{x} \varphi'(H(y))\, dF_{Nj}(y)$$

Noting that $\varphi(H(x)) = \int_{x_0}^{x} \varphi'(H(y))\, dH(y) + \varphi(H(x_0))$, and setting

$$A_{Ni}(x) = \frac{c_{Ni}}{N} \sum_{j=1}^{N} \int_{x_0}^{x} \varphi'(H(y))\, dF_{Nj}(y) - \frac{1}{N} \sum_{j=1}^{N} c_{Nj} \int_{x_0}^{x} \varphi'(H(y))\, dF_{Nj}(y)$$

$$= \frac{1}{N} \sum_{j=1}^{N} (c_{Ni} - c_{Nj}) \int_{x_0}^{x} \varphi'(H(y))\, dF_{Nj}(y)$$

we obtain

$$\text{Var}\,(B_{1N} + B_{2N}) = \sum_{i=1}^{N} \text{var}\,(A_{Ni}(X_{Ni})) .$$

<u>Proof of Theorem</u> 1.2. Using (2.7), we have

(2.34) $$\frac{S_N - u_N}{s_N} = \frac{T_N}{s_N} + \frac{D_N}{s_N}$$

where

(2.35) $$T_N = B_{1N} + B_{2N}, \text{ and } D_N = D_{1N} + D_{2N} + D_{3N} .$$

Denote

(2.36) $$F_N(x) = P\left(\frac{S_N - u_N}{s_N} \le x\right), \; G_N(x) = P\left(\frac{T_N}{s_N} \le x\right)$$

Then

(2.37) $$F_N(x) - \Phi(x) = G_N\left(x - \frac{D_N}{s_N}\right) - \Phi(x)$$

$$= \left[G_N\left(x - \frac{D_N}{s_N}\right) - \Phi\left(x - \frac{D_N}{s_N}\right)\right] + \left[\Phi\left(x - \frac{D_N}{s_N}\right) - \Phi(x)\right]$$

Since (by Polya's theorem), both $F_N(x) - \Phi(x)$ and $G_N\left(x - \frac{D_N}{s_N}\right) - \Phi\left(x - \frac{D_N}{s_N}\right)$ converge to o uniformly in x , it follows that the random quantity

(2.38) $$\Delta_N = \sup_x \left|\Phi(x - \frac{D_N}{s_N}) - \Phi(x)\right| \text{ converges to zero.}$$

We now estimate

(2.39) $$\sup_x \left| G_N(x - \frac{D_N}{s_N}) - \Phi(x - \frac{D_N}{s_N})\right| = \sup_x \left| G_N(x) - \Phi(x)\right|$$

Observe that by the Berry-Esseen theorem (cf. Feller (1971), page 544),

(2.40) $$\sup_x \left| G_N(x) - \Phi(x)\right| \le \frac{C\rho_N^3}{s_N^3}$$

where C can be taken to be .7975 (cf. Van Beek (1972), and Bhattacharya-Ranga Rao (1976)).

From (2.40) and (2.37), we have

(2.41) $$\sup_x \left| F_N(x) - F(x)\right| \le \frac{C\rho_N^3}{s_N^3} + \Delta_N$$

It remains to show that $N^{\frac{1}{2}}\Delta_N = O_p(\|\varphi'\| s_N^{-1} \sum_{i=1}^N |c_{Ni}|)$

From (2.38), since

(2.42) $$\left|\Phi(x - \frac{D_N}{s_N}) - \Phi(x)\right| = \left|\frac{D_N}{s_N}\right| \Phi'(x - \frac{\alpha D_N}{s_N}) \text{ for some } \alpha,\ 0 \le \alpha \le 1$$

$$\le \frac{1}{\sqrt{2\pi}} \left|\frac{D_N}{s_N}\right|$$

it suffices to show that $N^{\frac{1}{2}} D_N = O_p(\|\varphi'\| s_N^{-1} \sum_{i=1}^N |c_{Ni}|)$.

Now re-arranging the terms of D_N , we obtain

(2.43) $$D_N = \int_{-\infty}^{\infty} \left\{\varphi\left(\frac{N}{N+1} H_N(x)\right) - \varphi(H(x))\right\} dC_N(x) - \int_{-\infty}^{\infty} (H_N(x) - H(x))\, dC(x)$$

To simplify the proof, we drop the factor $\frac{N}{N+1}$ since it does not affect the conclusion. By the mean - value theorem

(2.44) $$\varphi(H_N(x)) - \varphi(H(x)) = (H_N(x) - H(x))\, \varphi'(\xi_N(x)) \text{ for some } \xi_N(x).$$

Hence

$$\left|\frac{N^{\frac{1}{2}}}{s_N}\int_{-\infty}^{\infty}(H_N(x)-H(x))\,\varphi'(\xi_N(x))\,dC_N(x)\right| \le \frac{\|\varphi'\|}{s_N}\left|\int_{-\infty}^{\infty} N^{\frac{1}{2}}(H_N(x)-H(x))\,dC_N(x)\right|$$

Let $\varepsilon > 0$ be given. Then from Puri and Sen (1971), there exists a constant $C(\varepsilon)$ such that with probability $> 1-\varepsilon$,

$$\sup_x N^{\frac{1}{2}}\,|H_N(x)-H(x)| < C(\varepsilon)$$

Hence, with probability $> 1-\varepsilon$,

(2.45) $$\left|\frac{N^{\frac{1}{2}}}{s_N}\int_{-\infty}^{\infty}(H_N(x)-H(x))\,\varphi'(\xi_N(x))\,dC_N(x)\right| \le \frac{C(\varepsilon)}{s_N}\,\|\varphi'\|\sum_{i=1}^{N}|C_{Ni}|$$

The proof of

(2.46) $$\frac{1}{s_N}\left|\int_{-\infty}^{\infty} N^{\frac{1}{2}}(H_N(x)-H(x))\,\varphi'(H(x))\,dC(x)\right| \le \frac{C(\varepsilon)}{s_N}\,\|\varphi'\|\sum_{i=1}^{N}|C_{Ni}|$$

in probability is identical.

(2.45) and (2.46) establish the theorem (1.2).

REFERENCES

1. Beek, P. van (1972). An application of the Fourier method to the problem of sharpening the Berry-Esseen inequality. Z. Wahrscheinlichkeits theorie verw. Geb. 23, 187-197.

2. Bergström, H. and Puri, M.L. (1977). Convergence and remainder terms in linear rank statistics. Ann. Statist. 5

3. Bhattacharya, R.N. and Ranga Rao, R. (1976). Normal approximation and asymptotic expansions. John Wiley, New York.

4. Bickel, P.J. (1974). Edge worth expansions in nonparametric statistics. Ann. Statist. 2 , 1-20 .

5. Chernoff, H. and Savage, I.R.(1958). Asymptotic normality and efficiency of certain nonparametric test statistics. Ann. Math. Statist. 29, 972-996.

6. Feller, W. (1971). An introduction to Probability theory and its applications, vol. 2. John Wiley, New York.

7. Hájek, J.(1968). Asymptotic normality of simple linear rank statistics under alternatives. Ann. Math. Statist. 39, 325-346.

8. Hoeffding, W. (1973). On the centering of a simple linear rank statistics. Ann. Statist. 1, 56-66.

9. Jurečkova, J. and Puri, M.L.(1975). Order of normal approximation for rank test statistics distribution. Ann. Prob. 3, 526-533.

10. Loève, M. (1963). Probability Theory, 3rd ed. Van Nostrand, New York.

11. Puri, M.L. and Sen, P.K. (1971). Non parametric methods in multivariate analysis. John Wiley, New York.

12. Stein, C. (1970). A bound for error in the normal approximation to the distribution of a sum of dependent random variables. Proc. 6th Berkeley Symp. Math. Stat. and Prob. vol. 2 .

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFOSR - TR - 77 - 0451	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) ASYMPTOTIC NORMALITY AND CONVERGENCE RATES OF LINEAR RANK STATISTICS UNDER ALTERNATIVES,		5. TYPE OF REPORT & PERIOD COVERED Interim rept.
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Madan L. Puri and Navaratna S. Rajaram		8. CONTRACT OR GRANT NUMBER(s) AF-AFOSR 76-2927-76
9. PERFORMING ORGANIZATION NAME AND ADDRESS Indiana University Department of Mathematics Bloomington, IN 47401		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F 2304/A5
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research/NM Bolling AFB, Washington, DC 20332		12. REPORT DATE 1977
		13. NUMBER OF PAGES 15
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) 17p.		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

linear rank statistics; convergence rates; score generating function

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Asymptotic normality of linear rank statistics under alternatives is proved employing techniques of Chernoff and Savage (1958 Ann. Math. Statist., 29, 972-994). Under suitable assumptions the rate of convergence to normality under alternatives is also obtained for such statistics. The results of the paper are related to those of Hájek (1968, Ann. Math. Statist., 39, 325-346), and

402 522

20 Abstract

Hoeffding (1973, Ann. Statist., 1, 54-66). Results on the rate of convergence extend those of Jurečkova and Puri (1975, Ann. Prob., 526-533), and Bergström and Puri (1977, Ann. Statist.).